For a continued revival of the philosophy of mathematics

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1 Introduction

I would like to begin this talk by paying homage to Roshdi Rashed. The talk is intended as a friendly and grateful tribute to him. It is therefore appropriate that I should talk on issues concerning philosophy of maths and particularly of recent mathematics. As a Historian of mathematics he is a tireless reader of contemporary mathematics (like the great Neugebauer was also). He knows how to draw, for example from Category Theory, examples and ways of thinking that serve as benchmarks for exploring the conceptual history of mathematics

1.1 Mathematics

I would like to begin by mentioning some reflexions by Alain Connes. He remarks that

> *Mathematics is the backbone of modern science and a remarkably efficient source of new concepts and tools to understand the "reality" in which we participate. It plays a basic role in the great new theories of physics of the XX^{th} century such as GR and QM. The nature and inner workings of this mental activity are often misunderstood or simply ignored even among scientists from other disciplines. They usually only make use of rudimentary mathematical tools that were already known in the XIX^{th} century and miss completely the strength and depth of the constant evolution of our mathematical concepts.*

These assertions are still more true of the philosophers. While it is true that there are exceptions like Cavaillès, Lautman, and some of the historians of mathematics, nevertheless one can say that the living heart of the activity of mathematics in action is generally ignored.

My purpose is to provide some elements which can help begin to transform this situation. I can propose only a modest contribution in the face of the immense task which needs to be undertaken.

1.2 Some essential feature of the mathematical landscape

At first glance this landscape seems immense and diversified : it appears to be a union of separate parts like Geometry, Algebra, Analysis, Number Theory etc. Some parts are dominated by (the various aspects of) our understanding of the concept of "Space", others by the art of manipulating symbols", and others by the problems posed in our thinking about "infinity" and "the continuum".
This first view is not completely false, but this breaking down of mathematics into different regions of inquiry also misses much - it has a superficial aspect and needs to be rectified and re-elaborated by being brought together with other elements.

1.2.1 The unity of mathematics

The most essential feature of the mathematical world is the following : it is virtually impossible to isolate any of the above parts from the others without depriving them of their essence (Alain Connes). In order to describe this profound unity of mathematics we must take into account the nature of mathematical abstraction, and the manner in which it is related to the unity of maths.
On this view the essence of mathematics is linked to its unity. The first way to think of this unity is to compare the mathematical corpus with a biological entity : it can function and flourish only as a whole and would perish if separated into disjoint pieces. There are many ways to think of this organic metaphor for the unity of mathematics. I would like to emphasize four.

Firstly : it is a very old view in the history of science. It is a conception whose scope is universal and which serves to order our understanding not only of mathematics but of the manner in which the whole physical or biological world is to be thought of as a unity. Plato, for instance, insisted on the organic unity of geometry with respect to a hierarchy of ideal entities (the Forms) and hence of the Universe and Penrose to give another example, in his construction of the geometry of the Universe and more specifically in his reflexion on the role of complex geometry and his hypothesis that the key to the unity of nature lies in complex geometry.

Secondly: as an organic unity, it develops from inside, just as a living being. I will say more about this feature below

Third: this growth can take a variety of directions which carry simultaneous and multiple meanings and exhibit, so to speak, different rhythms of development.

Fourth: the "topologies" of these different kinds of increase can take very different forms, including - metaphorically speaking - very non-classical topological spaces of representation.

1.3 Internal endogenic growth

An essential feature of this organic development of mathematics is the extension of the corpus by new elements emerging from within, as in a living being. A theory may typically provide the resources for the expression of a further theory which develops by means of a *reflection* on the first. I'll add some elements of analysis to develop this topic below. Even for the calculus that Newton and Leibniz in different ways invented, it is only when it became a question for the mathematical corpus, when it adapted itself to a host structure produced by the corpus that one arrived at "the" calculus. The absorption of a notion arising from physical reflection by the mathematical corpus resulted in the production of a purely mathematical concept. This slow process of transformation of Euclidean concepts of motion has been studied by Marco Panza [P]

Any apparently external element, object, idea, image must be integrated and re-constructed in a mathematical manner and form. It is not certain that the calculus could have appeared without the intervention of physics, but the physical question had to be entirely transformed, mathematized) conceived as a mathematical problem in the passage from its initial Euclidean setting to the analytic one.

Moreover there are some difficulties with the organic metaphor. This metaphor misses a central aspect that characterizes mathematics. The fact that different disciplines have appeared which are at each time essential for all existing mathematics. For example topology, or algebraic geometry $\cdots$. Topology has spread to all mathematics and helped to renew old theories and to approach them in a new light. Each discipline has effects on others in various ways. Then a supplementary body appears to be essential for another. It is possible to follow the various ways a discipline (such as Algebra, Topology, Geometry $\cdots$) leaves the marks of its growth within the corpus. They can go through very different stages of growth and roles.

1.4 Two opposite concepts of the mathematical corpus

The Coming-to-be of mathematics appears as autonomous, unpredictable and endogenous and in accordance with a temporality such that the overall structure is out of reach. It typically involves bifurcations, branches, breaks, continuity, recovery, neighborhood relations, and moments of partial unification. We can try to propose two "extremal" forms of such development.

a) Labyrinthine

There exist many underground networks, Archimedes is related to Lebesgue and Riemann, but Archimèdes is also related to Pascal and Leibniz, Lagrange, Galois to Grothendieck. There are profound underground paths, sometimes surprising . It also happens that new proofs of the same theorem come as secondary benefit of a new theory. Reintroduction of the Pythagorean theorem in infinitesimal geometry renewed its sense. Multiple time-frames are sometimes involved in this.

b) Architectonic

These underground networks of conceptual connections can find expression in great "open"' theories which promise to re-shape the architectonic of mathematics : recent examples suggest themselves - scheme theory, descent theory, homotopy theory, topos theory.

I recall some of the principal new ideas Grothendieck consider as essential to his work [RS]

1.Tensor topological products and nuclear spaces

2. Continuous and discrete duality. (derived categories , "six operations").

3. Riemann-Roch-Grothendieck Yoga (K-theory, relation with intersection theory).

4. Schemes.

5. Topos.

6. Etale and l-adic Cohomology.

7. Motives and motivic Galois group (Grothendieck categories).

8. Crystal and crystalline cohomology, yoga of "de Rham coefficients ", "Hodge coefficients". . .

9. "Topological Algebra" : 1-stacks, derivators ; cohomological topos formalism, as inspiration for a new homotopic algebra.

10. Tame Topology.

11. Algebraic anabelian geometry Yoga, Galois-Teichmüller theory.

12. Schematic or arithmetic point of view for regular polyhedras and regular configurations in all genus.

Each of these "new ideas" plunges deeply into the mathematical corpus and imposes on

it a new systematic unity, or at least re-shapes our perspective on the different forms of unity it exhibits and enables us to trace new connections between them. The fact that we can distinguish these two opposite conceptions is as such significant. They are two forms of the creative productivity of mathematics. The first is that form in which it escapes us. The second one is the form in which it gives ways of exercising control over its forms of expansion. We are able to recognise new trajectories and detect new relations, for example, the program of derived Algebraic Geometry, that consider polynomial equations up to homotopy. This is a new trajectory within a the program. More precisely, it is a combination of schema theory and homotopy theory. Schema theory is re-worked from a homotopical perspective. The synthesis of both theories retains the power of each within a further unity. This allows a higher level viewpoint permitting us to reinterpret both theories and at the same time provides them with greater power.
This description can be repeated for every theme developed by Grothendieck.

> *Indeed, the countless questions, concepts , statements I just mentioned, do not take me for a meaning in the light of such a 'perspective " - rather, they arise spontaneously, with the strength of the evidence; in the same way as from light (even diffuse) which arises in the dark night, there seems to be born out of nothing these more or less blurred or sharp edges that it suddenly reveals. Without this light that unites them in a common beam, ten or a hundred or a thousand questions, concepts, statements appear as a heterogeneous and amorphous heap of " mental gadgets", insulated from each other.* [AC]

2 This unity took different forms in the history of mathematics

There at least two other ways to think of this unity[1]. The first is founded on logical developments like those proposed by Russell, Carnap and Wittgenstein. Lautman condemned this approach as divorced from mathematical reality. The philosophy of mathematics he wanted to develop was intended as a positive study, resting on two different approaches. These are closely connected with the two other forms of unity I want to discuss here.
From the side of one of these, mathematical reality can be characterized by the way one apprehends and analyzes its organization. From the side of the other, it can also be characterized in a more intrinsic fashion, from the point of view of its own structure. In summary, we may speak of a contrast between characterisations from the outside or from the inside. The first case was illustrated by Hilbert's position. He stressed the dominant role

[1]In the contemporary corpus it is distinctive and doubtless particularly multiform.

of metamathematical notions compared to those of the mathematical notions they serve to formalize. On this view, a mathematical theory receives its value from the mathematical properties that embody its structure in some generic sense. We recognize in this approach one (very influential) structural conception of mathematics.

a) In this first, structural conception, mathematics is seen as - if not a completed whole then at least as a whole within which theories are to be regarded as qualitatively distinct and stable entities whose interrelationships can in principle be thought of as completely specifiable,

b) In the second conception we recognize a more dynamic picture of the interrelationships which sees each theory as coming with an indefinite power of expansion beyond its limits bringing connections with others, of a kind which confirms the unity of mathematics, especially from the standpoint of mathematical epistemology.

In Hilbert's metamathematics one aims to examine mathematical notions in terms of logical notions, in particular those of non-contradiction and completion. This ideal turned out to be unattainable. Lautman wanted to develop a framework which combines the fixity of logical concepts and the development that gives life theories.

2.1 Dialectics

Lautman (in a third conception) wanted to consider also other logical notions that may also be connected to each other in a mathematical theory such that solutions to the problems they pose can have an infinite number of degrees. In this picture Mathematics set out partial results, reconciliations stop halfway, theories are explored in a manner that looks like trial and error, which is organized thematically and which allows us to see the kind of emergent linkage between abstract ideas that Lautman calls *dialectical*.

Contemporary mathematics, in particular the developing relations between algebra group theory and topology appeared to Lautman to illustrate this second "Labyrinthine" model of the dynamic evolving unity of mathematics, structured around oppositions such as local/global, intrinsic/extrinsic, essence/existence. It is at the level of such oppositions that philosophy intervenes in an essential way

2.2 Philosophical choices

Lautman wants to study specific mathematical structures in the light of the above oppositions. In his book [L], there is a chapter about "Local/global". He studies the almost organic way that the parts are constrained to organize themselves into a whole and the whole to organize the parts. Lautman says "almost organic" : one thinks of this expression in the following way. There exists an "organic" unity within mathematics, reminiscent of biological systems, as Alain Connes amongst many others has noted. I want to develop this recognition further below by making use of the notion of reflexivity. In his chapter about extrinsic and intrinsic with title "Intrinsic properties and induced properties" Lautmann examines whether it is possible to reduce the relationships that some system maintains with an ambient medium to properties inherent to this system. In this case he appeals to classical theorems of algebraic topology. More well-known is his text on " an ascent to the absolute" in which an analysis of Galois theory, class field theory, and the uniformisation of algebraic functions on a Riemaniann surface is presented. Lautman wanted to show how opposite philosophical categories are incarnated in mathematical theories. Mathematical theories are data for the exploration of ideal realities in which this material is involved.

2.3 Platonism?

All these opposites appear to be presented as ideal categories that exist in some Platonic world. What is this Platonism? I can reply with Lautman in the following way : Lautmann does NOT admit the possibility of an unchanging universe of ideal mathematical beings. The properties of a mathematical "being" depend upon the axioms of the theory in which it is defined. That speaks against any conception of an immutable intelligible Universe. But Lautmann considers that numbers and geometrical figures possess an objectivity as surely as the physical reality the mind encounters in the observation of nature. This objectivity of mathematical beings is perceived by the senses in the complexity of nature and reveals its true meaning in a theory of the involvement of mathematics in a higher and more hidden reality that constitutes a world of ideas. This is something more than an experiment within an experiment.

2.4 Platonism versus constructivism

Cavaillès disagreed with Lautmans view. He resists imposing any philosophical viewpoint that would dominate or control the effective thinking of the mathematician, he sees the requirement in the problem itself but without dialectical insight into the structure of the problem as guidance one is left with only empty generalities. It seems to me that it is possible to meet these two requirements, by appealing to CT. See below. Cavaillès seeks in mathematics itself the philosophical sense of mathematical thinking. By contrast Lautman sees it in the connection of mathematical concepts to a prior dialectic of which they are extensions. Their positions are not so very different : constructivism in Cavaillès gives rise to philosophical categories. And these are also drawn from a pre-existing philosophical corpus.

3 Another choice is possible : basic unities

It is possible to follow the development of basic unities: number, space , curves etc. These basic unitary notions are often seen as poles that articulate and supply orientation with respect to the background against which mathematical work germinates.
It is also possible to analyze different forms of connection between the parts of the mathematical corpus or even different disciplines within the corpus. Each such analysis will reveal a philosophical choice on the issue of the sense of the notion of mathematical existence. In other words, this mean that an analysis of different forms of connection between parts of the mathematical corpus (eg geometry and logic) will reveal a different philosophical orientation with respect to the question of the "existence" of mathematical entities or structures - typically an Platonist or Anti-Platonist orientation. One aspect of this claim which I find appealing is that it recognises that an analysis of such connections rests on an implicit understanding of the very subtle relationships between epistemological, ontological and perhaps especially in mathematics also more internal, and partially autonomous methodological factors whose interrelationship controls our understanding of the multiple senses involved in the notion "is a foundation for" or "is grounded in" - in the first place here, obviously with respect to mathematical concepts - but the issues raised affect every area of inquiry. One problem is that in the analytic tradition, since Frege there has been a largely concealed and very deep presupposition that our notion of "foundation(s)" is already ordered more or less exhaustively by an autonomous and purely logico-semantic dimension which sets the very standard of what counts as a definition or an explanation. The problem with this presupposition is that prevenst our recognizing or facing the prior question - "Where Do Meanings Come From?" So we are forced to confront these choices. Both approaches, Cavaillès' s and Lautman's are practicable.

3.1 The concept of space

In seeing the issues involved in making this choice, it is instructive to study the history of the concept of space. This intersects with the history of several disciplines and the development of forms of *reflexivity*, that is the main topic of this talk. Mathematical activity involves, as an essential aspect, examining concepts, theories or structures through (the lens of) other concepts theories and structures which we recognise as reflecting them in some way.

There are many ways to understand the notion of reflexivity in mathematical practice. One such way is illustrated by the stacking of algebraic structures, groups, rings, fields, vector space, modules, etc. Each level is the extension of the previous one - a vector space for example is a certain kind of module. The extension here consists in adding a property or a law. This imposition of additional structure brings a new perspective on the initial structure. A second way involves the addition of some property coming from another domain altogether, as seen in the notions of topological group, Lie group, differential or topological field. This synthesis also yields a new view of the initial structure. This is reflexivity in the weak sense. The effect of such new syntheses makes up much of the history of mathematics. But one has synthesis also between structures or concepts.

This other kind of the reflexivity includes the case where one discipline, for example, algebra reflects some concepts or some structures from another discipline. For example in algebraic topology, algebraic concepts and methods are used to translate and to control some topological properties. The same holds for algebraic geometry. And it can happen that algebraic topology and geometry themselves cross-fertilize by means of such reflexive interactions. All the phenomena of translation of one discipline in to another also illustrate such forms of reflexivity. This is a local manifestation of the fact that mathematics is permeated with such “reflecting surfaces”.

It is possible also to construct the history of one concept, for example the concept of number or of space viewed from this standpoint. G. G. Granger, a French philosopher of mathematics, says that these concepts are “natural”. But they are also the most opaque.

Take the history of the concept of space. As already noted, this history involves the intersection of multiple disciplines and the development of multiple forms of *reflexivity*, that form the main topic of this talk.

3.2 The concept of manifold

In the case of space, there was a long process whereby a deepening reflection on the concept of surface was produced in mathematics. Along the way, such a concept as that of variety was revealed. The concept of variety arose as a geometrical *reflection* on the concept of surface : First came the notion of an abstract surface parameterized by coordinates, then that of abstract place covered by topological opens (maps, atlas) in relation to an ambient space. These notions were extracted from such ‘reflexive” contexts as autonomous concepts which could be seen as defining a new kind of mathematical entity.

This extraction involved abstraction from the concept of surface, an abstraction which at the same time brought a change of point of view on the earlier concept : one passed from a concept defined via coordinates to one resting on parameters. That passage was effected by a reflection on the sense of using coordinates. One can understand that a surface is nothing but the different forms of the variation of its coordinates. And when one speaks in terms of maps and atlases the concept is further deeply reworked as was achieved by Hermann Weyl in his *Concept of Riemann surface*. This new entity now acts as the carrier of topological properties. And manifolds come to be seen as autonomous entities and indeed as a fundamental concept. The act whereby we obtained a surface is geometrically displaced, so to speak, and in this act of displacement the entity to which it is related is re-defined. The notion of a Variety is likewise designated in functional terms: it is the range of variation of the values of certain functions. Functions can now reflect their nature by means of this new entity. A manifold becomes the support and mirror for the properties of functions that are defined on it. These functions with their properties constitute the new objects we should consider as new basis and point of departure for a further stage of geometrization.

Thus our notion of space changes status, it becomes an intelligible object in itself and that is why it can provide a reflexive context in which to re conceptualize the previous notion of a surface. At the same time, the act of measuring can be considered as such and made the object of study as a structure within the mathematical corpus. The concept of a metric on a manifold makes possible this new reflexion. Any such expression of magnitude can be reduced to a quadratic form and thereby expresses the most general law that defines the distance between two infinitely near points of a variety.

This entity in turn enables us to construct new spaces : we can now define the notions of algebraic manifold, topological manifold, differentiable manifold, analytic manifold, arithmetic manifold. In this way we are given the means to pass from one discipline to another. This passage between formerly separated disciplines involved both an upward movement (in the formation of the concept of manifold) and horizontal and synthetic extension of concepts (across several domains and disciplines).

3.3 Some mathematical representations

Starting from the concept of manifold, we can develop the concept as support for the tools one use to explore the properties of the concept itself. A topology on a set X serves to define the continuous functions from the space X to the reals **R** or from any open U to **R**.

The continuity of each $f : U \to \mathbf{R}$ can be determined locally.

That means

i) If $f : U \to \mathbf{R}$ is continuous and $V \subset U$ is open, then the function f restricted to V $f_{|V} : V \to \mathbf{R}$ is continuous.

ii) If U is covered by open sets U_i and the functions $f_i : U_i \to \mathbf{R}$ are continuous for all $i \in I$, then there is at most one continuous function $f : U \to \mathbf{R}$ with restriction $f_{|U_i} = f_i$ for all i.

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Moreover such an f exists if and only if the various given f_i match together on all the overlaps $U_i \cap U_j$. In the sense that $f_i x = f_j x$ for all $x \in U_i \cap U_j$ and $i, j, \in I$. The second property states that continuous functions are uniquely *collatable*.

3.4 Philosophical meaning of this mathematical concept

A first extension process starts from a reflexion on the functions that define the manifold. And then by constructing another concept that determine these functions. Indeed the properties i) and ii) can be conveniently expressed in terms of the function C which assigns to each open $U \subset X$ the set of all real valued real continuous functions on U

$$C(U) = CU = \{f | f : U \to \mathbf{R} \text{ continuous}\}$$

For $V \subset U$the operation of
(i) restricting each f to the subset V, written as $f \mapsto f_{|V}$is a function $CU \to CV$, while if $W \subset V \subset U$ are three nested opens sets, restriction is transitive in that $(f_{|V})_{|W} = f_{|w}$.
Here is the new level of reflexivity. These two statements mean that the assignments

$$U \mapsto CU,\ \{V \subset U\}, \mapsto \{CU \mapsto CV\ by\ f \mapsto f_{|V}\}$$

define a functor

$$C : O(X)^{op} \to \mathbf{Sets}$$

3.4.1 The reflexivity by CT and functor

I will characterize very briefly this mode of reflexivity. We get a category $O(X)$ the category of all subsets of X and the arrows are the inclusions. The category characterizes a domain or a structure specific for a discipline (topology for example) and the functor is a translator of structure from one category to another one. Topology can thus be reflected in algebra or more simply in the category of sets.

It is easy to recognize that. Here we see mathematics working on itself. We seek for ways in which structure from one area can be reflected in another which supplies a means of analysis for the first. This other structure thus becomes an instrument of analysis. This analysis or reflection is very specific. The examples that I will provide here are necessarily brief.

Generally speaking I want e.g. to see which kind of an algebraic structure can unify and control a collection of functions. I can speak of a ring of continuous functions, or of a vector space of functions. The properties of these structures then permit us to analyze the functions that become objects as elements of a vector space, or elements of a ring. Functions are thus installed in the field of algebra. This greatly increases their synthetic power.

3.4.2 descriptive Sketch

I would here like to distinguish two kinds of reflexivity.

First there is a so-called extensive reflexivity : when I can recognize some properties of concept by unifying its objects by means of an algebraic structure (continuous functions as a ring or vector space) or by means of further outside structure imposed on the domain of objects, (eg topology on matrix spaces). One can in this way better understand the behavior of this concept, and apply to it elementary ring operations, vector space operations, or topological operations

Secondly, there is so called intensive reflexivity : a concept (so to speak) “involutes” itself and takes itself for its object, - as examples think of a function of functions, or a (functional), matrix of matrices.

There is also the case of mixed reflexivity, when the reflexive abstraction is at the same time extensive and intensive. I present only two examples, but there are many. The first is the already cited concept of manifold. As such it reflects the concept of surface and the concept of parametrization of a surface. The most important property of a manifold lies in the fact that one it is independently of parametrization or on charts. It is concerned with a reflection on the acts that coordinate a surface. It is not a reflexivity of form because there not a reduplication of the concept itself but rather in its reflection the act is made independent of its former context of definition and original conceptual support.

The extensive feature of the concept of manifold is also specific. One can distinguish a manifold by means of a structure on the objects that define it (algebraic, topological, analytic etc.).
Our second example is more complicated : it is the variety of the modules of semi stable bundles on a algebraic variety. In the casez of this example it is possible to examine different levels of synthesis, and to sees how the a higher level of conceptual synthesis operates on the concepts defined at a subordinate level. This remodeled concept is then at our disposal to play its part in further developments. Thus mathematics is built: a synthetic unity is constructed, which may in turn form part of a further dynamic synthesis. A first task of philosophy of mathematics has to be to classify the different types of synthesis involved in this evolution, depending on the nature of the concepts that provide the rules governing these syntheses.
In the case of the notion of a sheaf the situation is still more elaborate and subtle. A reflexive level is first extracted which describes and controls a range of objects. Here they are topological spaces and continuous functions. At this level they are then transferred to an algebraic structure.

3.5 Reflexivity and diversification of level of objects

One more question arises. These functions we have collected through an algebraic concept can they be defined on any topological space on a portion of which they take their values? Can we globalize this structure as such?
This is a question which algebraic geometry wants to answer. A new level of questions and thoughts is reached that opens a new field: that of reflection on the relationship between global and local. This is a new theme, a problem for domain objects with which we deal. This new meta level has a philosophical meaning which philosophy has to deal with.

The mathematical corpus is stratified, and thus is "typified" and therefore inaccessible to a set theory or at least to a classical set theory. The levels of this stratification are themselves diversified. CT has thematized this diversity of levels of reflexivity. I gave the example of the functor of a pre-sheaf. These operations of translation (functoriality), involving passage from one category to another are expressed by means of the concept of a functor. A functor preserves some properties of a category and translates them from one category to another. It is possible to rewrite much of the history of maths e.g. algebra through the study of this kind of translation (Galois theory e.g.). Indeed it allows us to understand that this history progresses via rising levels of reflexivity reflexivity. I can for example, express the system of roots of a polynomial by showing it corresponds to an field extension. By appealing to the concept of a field I in turn gain a new understanding of the role played by the groups

By means of functoriality I am given a new way to introduce and characterize the objects of a mathematical domain. That characterization typically brings consideration of the dialectic passage between local/global properties of a structure. Indeed the sheaf functor gives the means to precisely to "measure" the possibility of this passage. Viewed in another way this concept synthesizes the old question of continuity, of continuation of the solutions of DE. This trajectory of progressive synthesis and modification of concepts passes through various layers of stratification. In this way we get a new transversal synthesis. At the same time we secure an analysis which gives new significance to the concept of continuity significance, and indeed provides it with new meanings.

Mathematics reflects various forms of extension or passage from a local situation to a global situation (and vice-versa) and this reflection has been extracted (in the work of Leray and his successors) in the form of a concept - that of a sheaf - that supports this situation as such. And this concept can in turn be used to analyze many analogous situations. These different representations of representation (Kant) - or perhaps better, different ideas of an idea - is the main feature of such an extension. These two expression refer to the definition of concepts by Kant in *Critic of Pure Reason* as to the first, and to similar remarks concerning properties of ideas by Spinoza in the case of the second.

This upward and lateral and diagonal extension is the hallmark of mathematical development and it can be deepened in several directions through a study of the geography of these extensions. I give the example of Riemann surfaces. It is still a form of distancing : we reflect on the concept of Riemann surfaces. It was only possible to isolate the concept and provide a representation by deploying helical leaves above a ground plane. One thus had at one's disposal the means to give a geometrical meaning to functions of complex variable(s) like $\sqrt{z}$ It was partly inspiration by Riemann's idea of surfaces stacked over the plane that led Grothendieck to replace the open spaces of a space X by spaces stacked over it. This is in some ways a very typical mathematical production and invention, though in this case an exceptionally inspired and fruitful one. And it is just such theoretical movements that reflexivity (l'abstraction reflechissante) makes possible. The most important further topic, which unfortunately I have no time to develop here is the passage from the notion of an object to that of a morphism. This involves a kind of *a priori* summation of the reflexive point of view. Instead of considering one object included in another object I consider morphisms whereby the first one is actively included in the second.

The same thing can be expressed by considering the category $\mathfrak{F}(X)$ of sheaf on X. An additional stage of abstraction consists in considering sheaves as themselves the objects of a category. By means of this step we produced an additional level of reflexivity. Of which kind is this further reflexivity? CT furnishes a formal and (conceptual) level of reflexivity. In the sense of extracting some form that is preserved from one category to another one, in such a way that one category is able to supply a characterization of the another one, up to and including CAT, the category of categories.

CT supplies a vantage point from which we can characterize a structure in the first place by saying how it transforms specifying the answer by means of suitable arrows. And characterization is available whatever the abstraction level of the object. Armed with this concept one can translate the properties of a topological space into the language of morphisms in the world of categories. All these properties translate into constructions on categories of sheaves.

4 Topos

By an additional stage of abstraction Grothendieck, followed by Lawvere and Tierney proposed the abstract concept of a "topos" For him this ,was the ultimate generalization of the concept of space. But the concept of a topos is sufficiently general for the category of "all" sets to constitue a topos. From this standpoint it became clear (see. Pierre Cartier) how artificial were the default assumptions which had made it customary to fit all of mathematics into the framework supplied by the particular topos of sets.

Grothendieck claimed *the right to transcribe mathematics into any topos whatever*. In this conviction, he posited a level of synthetic abstraction at which we can give a mathematical sense to numerous conceptual analogies between different disciplines and theories, and this is one important aspect the role played by CT, and of Grothendieck's method.

Brouwer and Heyting long since remarked that the rules of intuitionistic propositional calculus resemble the rules for manipulation of open sets. This became clearer in the theory of topos. In any topos $\mathfrak{T}$ there is a logical object Ω whose "elements" are the truth values of the topos. When $\mathfrak{T}$ is the topos of sets, one has the classical values (true/false), but when the topos $\mathfrak{T}$ is the topos of sheaves over a space X, the truth values correspond to the open sets of X.

One can see a topos as a place (here etymology borows from abstract topology) where we can do mathematics, a kind of *a priori* conceptual system providing the means to do mathematics. I should insist on the fact that this notion of place is topological in its genesis.

5 First conclusion of this part

The first (philosophical) issue concerns my use here of the concept of reflexivity. I mean by this the fact that one structure or theory provides the means to explore and to explain the properties of another one. This concept might be deemed too large, too open, too flexible. But in the sense I use it here , one can distinguish the topological properties of an algebraical structure and vice versa. This is one mode of reflexivity. In a further mode, one can also define the characteristics of a whole theory by means of another. Reflexivity can also yield a theme which connects and organizes a great portion of mathematics and of the interrelationship of mathematical "structures". We have already mentioned the opposition of global / local and according to Lautman, extrinsic / intrinsic, continuous / discrete etc.

These reflexivities are at the same time of course syntheses. Such syntheses are illustrated in Algebraic Topology, Algebraic Geometry, Differential Topology, Analytic Geometry, different Number Theories, Discrete Geometry, Categorical Logic and other parts of mathematics.

Let us consider the Chow theorem 1949 . Let S_n be the projective space of n dimensions over the field k of all complex numbers. A compact analytic variety in S_n is an algebraic variety. The question is : can we identify a compact analytic variety and an algebraic one? The answer is : yes. In the framework of scheme theory the general question arises of the comparison between schemes of finite type over $\mathbb{C}$ and their associate complex analytic spaces. This simplifies the question under which conditions an algebraic object can be identified with an analytic one. It gave rise to the famous theorem GAGA, géométrie algébrique, géométrie analytique (1956). We remark that it is just through situations involving such reflexivity that mathematics produces fruitful new "positive" concepts.

That production is made is possible through different mode of "reflexivity". But a mathematical concept carries the marks of its reflexive development. This particular conceptual dynamic is universal in essence. It is a universality we can recognise in the conceptions of both Spinoza and Dedekind.

I would like to insist on another form taken by this process of conceptual extension. This is illustrated by the relations between algebra and geometry. That interplay as such takes account of the reflexivity process, as guided by CT.

I restrict myself here to indicating another kind of translation from one theory to another that is more in accord with what I call a "substantial parallelism". When a theory grows it tends to absorb other parts of the mathematical corpus in a powerful synthesis. This double process of reflexion and of synthesis is the source of the "organic" unity of which we have speaking in mathematics. We can highlight aspects of this synthetic unity by following the development of the concept of space

Our notion of *space* in algebraic geometry has evolved considerably in the second half of the XXth century under the impulsion of specific problems such as the Weil conjectures which were finally proved by Deligne in 1973. As A Connes summarizes after fundamental work by Serre who developed the theory of coherent sheaves (FAC) and a very flexible notion of algebraic variety, based on Leray's notion of sheaves and Cartan's notion of "ringed space", Grothendieck undertook the task of extending the whole theory to the framework of *schemes*. Schemes are obtained by patching together the geometric counter part of arbitrary commutative rings. We have here a notion of geometry "on rings". The "standard conjectures" of Grothendieck are a vast generalization of the Weil conjectures to arbitrary correspondences, whereas Weil's treatment is confined to a specific setting known as the "Frobenius correspondence".

Without developing the analysis in detail, one can remark that all these concepts arose from a reflexion of a conceptual level which led in some specific direction. It seems to me that in the case of Grothendieck's scheme theory the reflexivity involved becomes more refined. One gains a unified understanding, in the same setting, of a topological space and a sheaf of functions on it. The algebraic part of the concept, or of the system of concepts, is considered in parallel with the topological part. The level of synthetic reflexivity is higher. For this reason I spoke of *parallel synthetic reflexivity*, but one must consider that it is this parallelism that allows the gain in level.

5.0.1 Parallel between algebra and geometry

There exists another parallel between algebra and geometry. The classical geometry I spoke about is described by means of the concept of manifold. If I consider its algebra of coordinates it is easy to see that if forms a commutative algebra, even a C* algebra. We observe the duality

$$\underline{GeometricSpace}|\underline{CommutativeAlgebra}$$

This duality as such is interesting and its analysis is fruitful. The point of departure of non commutative geometry is the existence of natural spaces playing an essential role both in mathematics and in physics but whose "algebra of coordinates" is no longer commutative. The Phase Space in QM is one obvious example, but there are are many others. One sees in this very important case the role played by algebra as the usual tools break down. This situation forces us to reject the set theoretical point of view.

Even though as a set such spaces have the cardinality of the continuum, it is impossible to distinguish their points by a finite (or even countable) set of explicit functions. Any explicit countable family of invariants fails to separate points and the *effective* cardinality is not the same as that of the continuum.

5.0.2 Non Commutative Geometry

We have here entered the realm of non commutative geometry. The general principle (formulated by Connes) that allows one to construct the algebra of coordinates on such quotient space $X = Y \ \sim$ is to replace the commutative algebra of functions on Y which are constant along the classes of equivalence relation $\sim$ by the *non commutative* convolution algebra of the equivalence relation so that the above duality gets extended as

$$\underline{GeometricQuotientSpace|NonCommutativeAlgebra}$$

The main feature of quantum mechanics is, as is well known, the non commutativity of observables. How can we accommodate this? Surprisingly mathematicians discover that we can speak about spaces without even mentioning them. The trick is to use algebraic objects and the surprise is that space can be reconstructed from them. In the language of mathematics, one has an equivalence of categories.

This is one of the main results behind *non commutative geometry*. If commutative algebras give rise to ordinary topological spaces (in the category of C^* algebra) what are the spaces corresponding to non commutative ones? Some of the machinery developed on and for topological spaces can be applied to their non commutative counterparts the C^* algebra).

Topos can be seen as a kind of universe in which one can interpret higher order logics and do mathematics. In what sense can one say that we have to deal here with a form of reflexivity (reflexiveness)? A topos is a category with some essential properties that make possible minimal operations (like multiplication e.g.,) or minimal relations (like associativity of maps e.g.) we need in the practice of maths. Reflexivity here lies in the minimality of conditions we can extract to go to the next level.

F. e. let us consider *topos of smooth spaces*. A universe which includes among its objects a line R, R^2, and so on through all the classical manifolds of differential geometry and more including infinitesimal spaces. In the universe of this topos, every object has a geometric structure and every function is continuously differentiable

6 What pushes a mathematician to be engaged in a mathematical adventure?

Mathematics is constantly pushed ahead of themselves. What is destiny of this indefinite movement. Whatever the state of the historical development of mathematics, what guided mathematical research is the presence of relational schemas to anticipatory value opening on an explanation or invention of new relations in the thematic area under investigation. We use here the expressions of Maurice Caveing. This scheme means the ideal form of any matchmaking operation or anticipatory form $R_{x,y\ ,z}$ ...This is not the schema in the Kantian sense, i.e. as the generating image rule in intuition. It is more in the sens Mr. Caveing's formulation, as I remembered, and which is try to show that on the one hand the development of mathematics is relatively unpredictable, and yet they work on anticipatory resources available. By tapping into the resources of the relational thinking they are part of an unlimited development.
Those relational schemas are organized today in large stratified programs which are organized in a hierarchical form that expresses reflection levels on each other. It is by this means that we enter subjectively in the mathematical universe. Because this is where our thinking is unified with already given thought.
Often an act of rebellion of a young mathematician against the existing mathematical world is the starting point. Once he comes to know a small part of the mathematical world, through a personal and original way, he can begin his journey [A C] within this relational world in which he just made his first experience. The entrance into this world is a personal act, subjective and at the same time very constrained. This dual nature of subjective freedom and totally constrained define the dual nature of reflexivity or even mathematical thinking. It is only in this context that we can try to develop analyzes starting from the notion of epistemic intuition, as G Heinzmann did [GH]. The epistemic intuition is a form of intuition framed by the formality by which it acquires objectivity. A synthesis can then be accomplished on very different objects. G. G. Granger spoke here about formal intuition. The intuition areas are then found displaced by its partially guide formalism.

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